1 The ADM Decomposition

Consider a d-dimensional spacetime with metric h_{ab}. We start by identifying a scalar field t whose isosurfaces Σ_t are normal to the timelike unit vector given by

$$u_a = -\alpha\,\partial_a t \ , \tag{1}$$

where the lapse function α is

$$\alpha := \frac{1}{\sqrt{-h^{ab}\,\partial_a t\,\partial_b t}} \ . \tag{2}$$

An observer whose worldline is tangent to u_a experiences an acceleration given by the vector

$$a_b = u^c \cdot {}^d\nabla_c u_b \ , \tag{3}$$

which is orthogonal to u_a. The (spatial) metric on the $d-1$ dimensional surface Σ_t is given by

$$\sigma_{ab} = h_{ab} + u_a u_b \ . \tag{4}$$

The intrinsic Ricci tensor built from this metric is denoted by $\mathcal{R}_{ab}$, and its Ricci scalar is $\mathcal{R}$. The covariant derivative on Σ_t is defined in terms of the d dimensional covariant derivative as

$$D_a V_b := \sigma_a{}^c \sigma_b{}^e \left({}^d\nabla_c V_e\right) \quad \text{for any} \quad V_b = \sigma_b{}^c V_c \ . \tag{5}$$

The extrinsic curvature of Σ_t embedded in the ambient d dimensional spacetime (the constant r surfaces from the previous section) is

$$\theta_{ab} := -\sigma_a{}^c \sigma_b{}^d \left({}^d\nabla_c u_d\right) = -{}^d\nabla_a u_b - u_a a_b = -\frac{1}{2}\,\pounds_u \sigma_{ab} \ . \tag{6}$$

This definition has a minus sign relative to the definition I normally use. This is for compatibility with the standard conventions in the literature.

Now we consider a ‘time flow’ vector field t^a, which satisfies the condition

$$t^a\,\partial_a t = 1 \ . \tag{7}$$

The vector t^a can be decomposed into parts normal and along Σ_t as

$$t^a = \alpha\, u^a + \beta^a \ , \tag{8}$$

where α is the lapse function (2) and $\beta^a := \sigma^a{}_b t^b$ is the shift vector. An important result in the derivations that follow relates the Lie derivative of a scalar or spatial tensor (one that is orthogonal to u^a in all of its indices) along the time flow vector field, to Lie derivatives along u^a and β^a. Let S be a scalar. Then

$$\pounds_t S = \pounds_{\alpha\,u} S + \pounds_\beta S = \alpha \pounds_u S + \pounds_\beta S \ . \tag{9}$$

Rearranging this expression then gives

$$\pounds_u S = \frac{1}{\alpha}\left(\pounds_t S - \pounds_\beta S\right) \ . \tag{10}$$

Similarly, for a spatial tensor with all lower indices we have

$$\pounds_t W_{a\ldots} = \alpha \pounds_u W_{a\ldots} + \pounds_\beta W_{a\ldots} \ . \tag{11}$$

This is not the case when the tensor has any of its indices raised. In a moment, these identities will allow us to express certain Lie derivatives along u^a in terms of regular time derivatives and Lie derivatives along the shift vector β^a.

Next, we construct the coordinate system that we will use for the decomposition of the equations of motion. The adapted coordinates (t, x^i) are defined by

$$\partial_t x^a := t^a \ . \tag{12}$$

The x^i are d dimensional coordinates along the surface Σ_t. If we define

$$P_i{}^a := \frac{\partial x^a}{\partial x^i} \ , \tag{13}$$

then it follows from the definition of the coordinates that $P_i{}^a \partial_a t = 0$ and we can use $P_i{}^a$ to project tensors onto Σ_t. For example, in the adapted coordinates the spatial metric, extrinsic curvature, and acceleration and shift vectors are

$$\sigma_{ij} = P_i{}^a P_j{}^b \sigma_{ab} \tag{14}$$
$$\theta_{ij} = P_i{}^a P_j{}^b \theta_{ab} \tag{15}$$
$$a_j = P_j{}^b a_b \tag{16}$$
$$\beta_i = P_i{}^a \beta_a = P_i{}^a t_a \ . \tag{17}$$

The line element in the adapted coordinates takes a familiar form:

$$h_{ab} dx^a dx^b = h_{ab} \left(\frac{\partial x^a}{\partial t} dt + \frac{\partial x^a}{\partial x^i} dx^i \right) \left(\frac{\partial x^b}{\partial t} dt + \frac{\partial x^b}{\partial x^j} dx^j \right) \tag{18}$$
$$= h_{ab} \big(t^a dt + P_i{}^a dx^i \big) \big(t^b dt + P_j{}^b dx^j \big) \tag{19}$$
$$= t^a t_a dt^2 + 2 t_a dt P_i{}^a dx^i + h_{ab} P_i{}^a P_j{}^b dx^i dx^j \tag{20}$$
$$= \big(-\alpha^2 + \beta^i \beta_i \big) dt^2 + 2\beta_i dt dx^i + \sigma_{ij} dx^i dx^j \tag{21}$$
$$\Rightarrow h_{ab} dx^a dx^b = -\alpha^2 dt^2 + \sigma_{ij} \big(dx^i + \beta^i dt \big) \big(dx^j + \beta^j dt \big) \ . \tag{22}$$

Thus, in the adapted coordinate system we can express the components of the (d dimensional) metric h_{ab} and its inverse h^{ab} as

$$h_{ab} = \left(\begin{array}{c|c} -\alpha^2 + \beta^i \beta_i & \sigma_{ij} \beta^j \\ \hline \sigma_{ij} \beta^j & \sigma_{ij} \end{array} \right) \tag{23}$$
$$h^{ab} = \left(\begin{array}{c|c} -\frac{1}{\alpha^2} & \frac{1}{\alpha^2} \beta^i \\ \hline \frac{1}{\alpha^2} \beta^i & \sigma^{ij} - \frac{1}{\alpha^2} \beta^i \beta^j \end{array} \right) \tag{24}$$
$$\det(h_{ab}) = -\alpha^2 \det(\sigma_{ij}) \tag{25}$$

Obtaining the components of the inverse is a short algebraic calculation. Note that the spatial indices '$i, j, \ldots$' in the adapted coordinates are lowered and raised using the spatial metric σ_{ij} and its inverse σ^{ij}.

In adapted coordinates there are several results concerning the projections of Lie derivatives of scalars and tensors which will be important in what follows. The first, which is trivial, is that the Lie derivative of a scalar S along the time-flow vector t^a is just the regular time-derivative

$$\pounds_t S = t^a \partial_a S = \frac{\partial x^a}{\partial t} \frac{\partial S}{\partial x^a} = \partial_t S \ . \tag{26}$$

Next, we consider the projector $P_i{}^a$ applied to the Lie derivative along t^a of a general vector W_a, which gives

$$P_i{}^a \pounds_t W_a = \partial_t W_a \quad \forall \quad W_a \ . \tag{27}$$

The important point is that this applies not just to spatial vectors but to *any* vector W_a, as a consequence of the result

$$P_i{}^a \pounds_t u_a = 0 \ . \tag{28}$$

Finally, we can show that the Lie derivative along t^a of any contravariant spatial vector satisfies

$$P^i{}_a \pounds_t V^a = \partial_t V^i \quad \forall \quad V^i = P^i{}_a V^a \ . \tag{29}$$

This follows from a lengthier calculation than what is required for the first two results.

Given these results, we can express various geometric quantities and their projections normal to and along Σ_t in terms of quantities intrinsic to Σ_t and simple time derivatives. First, the extrinsic curvature is

$$\theta_{ij} = -\frac{1}{2} P_i{}^a P_j{}^b \pounds_u \sigma_{ab} \tag{30}$$

$$= -\frac{1}{2} P_i{}^a P_j{}^b \left(\frac{1}{\alpha} \left(\pounds_t \sigma_{ab} - \pounds_\beta \sigma_{ab} \right) \right) \tag{31}$$

$$\Rightarrow \theta_{ij} = -\frac{1}{2\alpha} \Big(\partial_t \sigma_{ab} - \left(D_a \beta_b + D_b \beta_a \right) \Big) \ . \tag{32}$$

Since θ_{ab} is a spatial tensor, projections of its Lie derivative along u^a can be expressed in a similar manner

$$P_i{}^a P_j{}^b \pounds_u \theta_{ab} = \frac{1}{\alpha} \left(\partial_t \theta_{ab} - \pounds_\beta \theta_{ab} \right) \ . \tag{33}$$

Now we present the Gauss-Codazzi and related equations in adapted coordinates:

$$P_i{}^a P_j{}^b \left({}^d R_{ab} \right) = \mathcal{R}_{ij} + \theta \theta_{ij} - 2\theta_i{}^k \theta_{jk} - \frac{1}{\alpha} \left(\partial_t \theta_{ij} - \pounds_\beta \theta_{ij} \right) - \frac{1}{\alpha} D_i D_j \alpha \tag{34}$$

$$P_i{}^a \left({}^d R_{ab} u^b \right) = D_i \theta - D^j \theta_{ij} \tag{35}$$

$${}^d R_{ab} u^a u^b = \frac{1}{\alpha} \left(\partial_t \theta - \beta^i \partial_i \theta \right) - \theta^{ij} \theta_{ij} + \frac{1}{\alpha} D_i D^i \alpha \tag{36}$$

$${}^d R = \mathcal{R} + \theta^2 + \theta^{ij} \theta_{ij} - \frac{2}{\alpha} \left(\partial_t \theta - \beta^i \partial_i \theta \right) - \frac{2}{\alpha} D_i D^i \alpha \ . \tag{37}$$

2 Converting to ADM Variables

The metric is often presented in the form

$$h_{ab} dx^a dx^b = h_{tt} dt^2 + 2h_{ti} dt dx^i + h_{ij} dx^i dx^j \ . \tag{38}$$

We would like to relate these components to the ADM variables: the lapse function α, the shift vector β_i, and the spatial metric σ_{ij}. This is a fairly straightforward exercise in linear algebra. Comparing with (22), we first note that

$$\sigma_{ij} = h_{ij} \ . \tag{39}$$

The inverse spatial metric, σ^{ij}, is literally the inverse of h_{ij}, which is *not* the same thing as h^{ij}

$$\sigma^{ij} = (\sigma_{ij})^{-1} = (h_{ij})^{-1} \neq h^{ij} \ . \tag{40}$$

For the shift vector we have

$$h_{ti} = \sigma_{ij}\beta^j \quad \rightarrow \quad \sigma^{ik}h_{tk} = \sigma^{ik}\sigma_{kl}\beta^l = \beta^i \tag{41}$$

$$\Rightarrow \beta^i = \sigma^{ij}h_{tj} \ . \tag{42}$$

Finally, for the lapse we obtain

$$\alpha^2 = \sigma^{ij}h_{ti}h_{tj} - h_{tt} \ . \tag{43}$$